A Semi-Closed Form of the Minimal Number of Simple Transpositions to Obtain the Identity Permutation Group Function

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Abstract

This paper will introduce the Parity Test function, which will assist in deriving a semi-closed form of the function for the minimal number of simple transpositions needed to apply to a permutation group to obtain the identity group.

Keywords

simple, transposition, identity, permutation, group, parity, positive, negative

Introduction

A permutation group is a group G whose elements are permutations of a given set M and whose group operation is the composition of permutations in G[1]. This can be written as:

$$\sigma = \begin{pmatrix} i & i+1 & \dots & r \\ \sigma(i) & \sigma(i+1) & \dots & \sigma(r) \end{pmatrix} \tag{1}$$

For simplicity of this paper, we will assume $M = \{1, 2, \dots r\}$ where $i, r \in \mathbb{N}^+$.

A transposition is where the image of two elements in the permutation group are interchanged, which can be written in cycle notation as (i, r), assuming the images of the i^{th} and the r^{th} elements are interchanged. The result of this transposition is:

$$\sigma = \begin{pmatrix} i & i+1 & \dots & r \\ \sigma(r) & \sigma(i+1) & \dots & \sigma(i) \end{pmatrix} \tag{2}$$

A simple transposition is a transposition where the two elements are consecutive, such as: $(i, i+1)$. For any permutation group, there is a minimal number of simple transpositions $n(\sigma)$ that can be applied to the permutation group to obtain the identity permutation group, which maps the elements to themselves.

We will prove the claim that the minimal number of simple transpositions function has the semi-closed form:

$$n(\sigma) = \frac{r(r-1)}{4} + \frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} \frac{|\sigma(j) - \sigma(i)|}{\sigma(j) - \sigma(i)} \tag{3}$$

Parity Test Function

To begin, let us derive the Parity Test function, which is a comprised of two Parity Test functions: the Positive Parity Test function (which allows us to determine whether or not x in $\mathbb{R}$ is positive) and the Negative Parity Test function (which allows us to determine whether or not x in $\mathbb{R}$ is negative). It is well-known in binary that a bit equal to 1 indicates yes and a bit equal to 0 indicates no, so we will adopt this convention[2]. Let us first look at the Positive Parity Test function. We are essentially looking for a function that computes:

$$f_+\left(x\right) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \tag{4}$$

This is a piecewise function, which can be unpleasant at times. Using the absolute value function, we can transform this into a function that is not piecewise. We know that $|x| \geq 0$ for all x in $\mathbb{R}$. Now, let us look at both cases when $x \leq 0$ and $x > 0$. We obtain a new piecewise function:

$$f_+\left(x + |x|\right) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases} \tag{5}$$

However, we wish for this to be equal to equation (4). We can achieve this by dividing by $2x$. Thus, we now have a function that is not piecewise to replace equation (4), which is the Positive Parity Test function:

$$P_+\left(x\right) = \frac{x + |x|}{2x} \tag{6}$$

We can follow the same process for the Negative Parity Test function. We are essentially looking for a function that computes:

$$f_-\left(x\right) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases} \tag{7}$$

Again, we know that $|x| \geq 0$ for all x in $\mathbb{R}$. Now, let us look at both cases when $x \geq 0$ and $x < 0$. We obtain a new piecewise function:

$$f_-\left(x - |x|\right) = \begin{cases} 0 & x \geq 0 \\ -2x & x < 0 \end{cases} \tag{8}$$

We wish for this to be equal to equation (7). We can achieve this by dividing by $2x$. Thus, we now have a function that is not piecewise to replace equation (7), which is the Negative Parity Test function:

$$P_-\left(x\right) = \frac{x - |x|}{2x} \tag{9}$$

Combining equations (6) and (9), we have the Parity Test function:

$$P_{\pm}\left(x\right) = \frac{x \pm |x|}{2x} \tag{10}$$

The Parity Test function, however, fails when $x = 0$ because of the division by zero. Trivially though, we know that zero is neither positive nor negative. Thus, we can state that $P_{\pm}\left(0\right) = 0$.

In the next section, we will discover how the Parity Test function helps in determining a semi-closed form of the minimal number of simple transpositions function.

The Minimal Number of Simple Transpositions

If a given permutation group is not the identity permutation group, simple transpositions can be applied to it to transform it into the identity permutation group. If we start from the image of the first element and compare it to the image of the next element, we can determine whether or not the two images need to be transposed. Recall our set M has all elements in $\mathbb{N}^+$. Then, we have a piecewise function for the minimal number of simple transpositions:

$$n(\sigma) = \begin{cases} n(\sigma) + 1 & \sigma(1) > \sigma(2) \\ n(\sigma) & \sigma(1) < \sigma(2) \end{cases} \tag{11}$$

The function starts with $n(\sigma) = 0$, which is also the minimal number of simple transpositions for the identity permutation group. This piecewise function is recursive as we compare the image of the first element with the images of all of the elements of higher indices:

$$n(\sigma) = \begin{cases} n(\sigma) + 1 & \sigma(1) > \sigma(k+1) \\ n(\sigma) & \sigma(1) < \sigma(k+1) \end{cases} \tag{12}$$

where $i < k \leq r$ and $k \in \mathbb{N}^+$. Suppose that $\sigma(1) = r$. Then, it can clearly be seen that $n(\sigma) = r - 1$ for $\sigma(r) = r$. Now, this piecewise function is doubly recursive as $1 \leq i < r$ and $i \in \mathbb{N}^+$ as we must compare all sets of two images of elements:

$$n(\sigma) = \begin{cases} n(\sigma) + 1 & \sigma(i) > \sigma(i+k) \\ n(\sigma) & \sigma(i) < \sigma(i+k) \end{cases} \tag{13}$$

From this piecewise function, we can see that we only add 1 if $\sigma(i) > \sigma(i+k)$. We can rewrite this statement as a function:

$$f(\sigma) = \begin{cases} 1 & \sigma(i) > \sigma(i+k) \\ 0 & \sigma(i) < \sigma(i+k) \end{cases} \tag{14}$$

Furthermore:

$$f(\sigma) = \begin{cases} 1 & \sigma(i) - \sigma(i+k) > 0 \\ 0 & \sigma(i) - \sigma(i+k) < 0 \end{cases} \tag{15}$$

This is exactly the Positive Parity Test function. Therefore, we can replace this with the Positive Parity Test function:

$$P_+(\sigma) = \frac{\sigma(i) - \sigma(i+k) + |\sigma(i) - \sigma(i+k)|}{2(\sigma(i) - \sigma(i+k))} \tag{16}$$

And thus:

$$n(\sigma) = P_+(\sigma) = \frac{\sigma(i) - \sigma(i+k) + |\sigma(i) - \sigma(i+k)|}{2(\sigma(i) - \sigma(i+k))} \tag{17}$$

Now, let us transform this function into an algorithm using summations:

$$n(\sigma) = \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} \frac{\sigma(j) - \sigma(i) + |\sigma(j) - \sigma(i)|}{2(\sigma(j) - \sigma(i))} \tag{18}$$

where j, i are two consecutive elements in the permutation group. Since $i = j+1$, it follows that the upper bound for i must be r and the upper bound for j must be $r-1$. We can factor out $\frac{1}{2}$:

$$n(\sigma) = \frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} \frac{\sigma(j) - \sigma(i) + |\sigma(j) - \sigma(i)|}{\sigma(j) - \sigma(i)} \tag{19}$$

We can also separate this into two terms:

$$n(\sigma) = \frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} \frac{\sigma(j) - \sigma(i)}{\sigma(j) - \sigma(i)} + \frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} \frac{|\sigma(j) - \sigma(i)|}{\sigma(j) - \sigma(i)} \tag{20}$$

The first term can be simplified:

$$\frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} \frac{\sigma(j) - \sigma(i)}{\sigma(j) - \sigma(i)} = \frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} 1 \tag{21}$$

We can remove the inner summation by further simplification:

$$\frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r} 1 = \frac{1}{2} \sum_{j=1}^{r-1} (r - j) \tag{22}$$

This can be expanded:

$$\frac{1}{2} \sum_{j=1}^{r-1} (r - j) = \frac{1}{2} \sum_{j=1}^{r-1} r - \frac{1}{2} \sum_{j=1}^{r-1} j \tag{23}$$

The r can be factored out of the first term:

$$\frac{1}{2} \sum_{j=1}^{r-1} r = \frac{r}{2} \sum_{j=1}^{r-1} 1 \tag{24}$$

Using the same concept to obtain expression (22), we simplify expression (24):

$$\frac{r}{2} \sum_{j=1}^{r-1} 1 = \frac{r(r-1)}{2} \tag{25}$$

We can simplify the second term in expression (23) using Gauss' summation of consecutive natural numbers formula:

$$-\frac{1}{2}\sum_{j=1}^{r-1} j = -\frac{r\left(r-1\right)}{4} \tag{26}$$

Adding the expressions (25) and (26), we obtain a replacement for the first term in equation (20):

$$\frac{r\left(r-1\right)}{2} - \frac{r\left(r-1\right)}{4} = \frac{r\left(r-1\right)}{4} \tag{27}$$

Let us now focus on the second term in equation (20):

$$\frac{1}{2}\sum_{j=1}^{r-1}\sum_{i=j+1}^{r} \frac{\left|\sigma\left(j\right)-\sigma\left(i\right)\right|}{\sigma\left(j\right)-\sigma\left(i\right)} \tag{28}$$

If we are given a permutation group with r elements, there are $r!$ possible images for the permutation group. Furthermore, the minimal number of simple transpositions is not unique to a given r, as $0 \leq n\left(\sigma\right) \leq \frac{r(r-1)}{2}$ where $n\left(\sigma\right) \in \mathbb{N}$. Therefore, we cannot determine which of them corresponds to the given permutation group just by knowing r. Thus, we need to know all of the images in the permutation group. Hence, expression (28) cannot be simplified. We have now compiled the function by adding together expressions (27) and (28):

$$n\left(\sigma\right) = \frac{r\left(r-1\right)}{4} + \frac{1}{2}\sum_{j=1}^{r-1}\sum_{i=j+1}^{r} \frac{\left|\sigma\left(j\right)-\sigma\left(i\right)\right|}{\sigma\left(j\right)-\sigma\left(i\right)} \tag{29}$$

Since this function matches function (3), our claim has been proven. There exists a semi-closed form of the minimal number of simple transpositions to obtain the identity permutation group function, which is function (3)/(29).

References

1. Permutation group - Wikipedia
(https://en.wikipedia.org/wiki/Permutation_group)

2. Binary number - Wikipedia
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